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Technical Memorandum No. GW 263

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ROYAL AIRCRAFT ESTABLISHMENT, FARNBOROUGHPreliminary survey of principles in the  
guidance of Blue Streak

by

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SUMMARY

The note examines the guidance of a ballistic rocket starting from basic principles. The work shows that guidance should be viewed in terms of velocity rather than position, and develops the concept of reference velocity, the velocity towards which the actual missile velocity should be directed. Several definitions of reference velocity are shown to be substantially equivalent. Approximation by a linear function of position is shown valid over a region roughly 60 miles long and 30 miles high but the coordinates used in this approximation are curvilinear and not suitable for an inertia navigator. The note ends with brief descriptions of three ways in which a ballistic rocket may be guided.

LIST OF CONTENTS

	<u>Page</u>
1 Introduction	2
2 Guidance in azimuth	2
3 Guidance in range	4
4 Reference velocity	6
5 Some definitions of reference velocity	9
6 Linear and quadratic fits to reference velocity	14
7 Some possible guidance principles	19
8 Conclusions	20
Acknowledgement	21
Glossary	21
References	22
Distribution	22

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LIST OF APPENDICES

Appendix

First and Second Differentials of Reference Speed for Fixed  
Climb Angle

I

First and Second Differentials of Reference Velocity for Least  
Reference Speed

II

LIST OF ILLUSTRATIONS

Figure

Contour of error between linear approximation and accurate reference  
speed

1

Contour of error between quadratic approximation and accurate  
reference speed

2

1 Introduction

Work in references 1 - 3 has been concerned with the guidance accuracy required for Blue Streak. The next problem which arises is how the required accuracy may be achieved. As a preliminary step towards guidance proposals, it is the purpose of this note to clarify the aims of the guidance system, to provide a framework in terms of which any guidance system should be viewed.

The first part of the note examines the effect of applying a control acceleration to a ballistic missile, thereby causing a slight modification to its ballistic path. The investigation shows that the effect on the velocity is of prime concern leading to the suggestion that guidance should be viewed chiefly in terms of the velocity of the missile.

This leads to the concept of a reference velocity which may be defined as the velocity which the missile is desired to possess. Several different definitions are worked out in the neighbourhood of a standard cut-off position 100 n. miles high assuming that the missile travels 2500 n. miles over the earth surface between cut-off and impact. The calculations assume a spherical, non-rotating earth exerting a gravitational field due to a uniform sphere.

Finally the note considers the accuracy of linear and quadratic approximations to particular definitions of the reference velocity. These show that a fairly simple computer could be designed for computing the instant of motor cut-off. The particular approximations require to be fed with curvilinear co-ordinates which could be provided by ground-based radar navigation but not by a missile-borne inertia navigator.

2 Guidance in azimuth

2.1 During the boost phase of the flight of a ballistic rocket, errors will arise which will cause the individual missile to become dispersed from the position and velocity which a standard missile might be expected to attain. These errors can be corrected continuously by guidance and control to an extent which depends on how large the errors are and what sacrifice in performance is tolerable. Suppose that the missile has been boosted to a

velocity which is approximately correct in order that the missile shall subsequently hit the target after travelling over its ballistic trajectory. Such a state of affairs may exist after the main boost motor has been cut-off while final trimming adjustments are being made to the velocity by means of a lower thrust vernier motor.

If the errors remain uncorrected, the missile will eventually reach the ground at some distance from the target. Consider the error in line at the target, that is the distance of the impact point from the plane of the great circle passing through the standard cut-off point and the target. It has been shown in Reference 1 that the line component of the impact error (correct to first order of errors) is

$$E_l = A \delta y + B \delta v \quad (1)$$

where  $\delta y$  is the error in line of the present missile position

$\delta v$  is the error in the azimuth component of the missile velocity

and  $A$  and  $B$  are constants depending on the trajectory.

Take as examples of the errors to be corrected

$$\delta y = 5 \text{ n. miles} \quad \text{and} \quad \delta v = 50 \text{ ft/sec.}$$

Consider a typical trajectory such that the missile travels 2500 n. miles from a cut-off 100 n. miles high, with the climb angle chosen so as to require least velocity at cut-off. Then

$$A = 0.4963 \times 10^{-4} \text{ n. miles per foot}$$

$$B = 0.1515 \text{ n. miles per ft/sec.}$$

Equation (1) shows that the error in line to be corrected amounts to

$$E_l = 9.07 \text{ n. miles.}$$

2.2 Suppose it is required to correct the error in line by means of an acceleration  $f$  acting for a time  $\tau$  seconds. At the end of the period of acceleration, the line component of the impact error is

$$A (\delta y - \frac{1}{2} f \tau^2) + B (\delta v - f \tau)$$

assuming that the constants  $A$  and  $B$  will be almost unchanged, which is true provided that the control duration  $\tau$  is short compared with the total time of flight over the trajectory. Equating the final impact error to zero shows that

$$f = \frac{A \delta y + B \delta v}{\frac{1}{2} A \tau^2 + B \tau} = \frac{E_l}{\frac{1}{2} A \tau^2 + B \tau}.$$

This may be written

$$\frac{E_l}{f} = \frac{1}{2} A \tau^2 + B \tau \quad (2)$$

SECRET - DISCREET

Technical Memorandum No. GW 263

which may be expressed in words as the line error at the target which is corrected by unit acceleration for the duration of the vernier control  $\tau$  seconds.

Taking a duration  $\tau = 20$  seconds,

$$\frac{1}{2}A\tau^2 = 0.0098 \text{ n. miles per ft/sec}^2$$

$$B\tau = 3.03 \text{ n. miles per ft/sec}^2.$$

From the relative sizes of the two components of expression (2), it is apparent that the effect of the change of heading is more than three hundred times the effect of the change in position. If the control duration were longer, the ratio would be rather less pronounced but even for a control duration ten times longer (i.e. 200 seconds) it remains true that the change of heading is more important than the change of position. The conclusion is that guidance in azimuth should aim at achieving a certain heading of the missile, the heading being a slowly changing function of position.

In order to correct the errors quoted above, the missile would require a lateral acceleration of  $3 \text{ ft/sec}^2$  for 20 seconds.

### 3 Guidance in range

3.1 While an error in time of flight may cause an incidental error due to rotation of the earth, the size of this error is not expected to be important so that the time at which the target is reached is not of prime importance. Thus the only other error which is important is the error in range at the target. This is governed to a varying degree by four errors at cut-off; namely, errors in speed, climb angle, height and ground range. It has been shown in Ref.1 that the range component of the impact error may be written (correct to first order of errors)

$$E_r = A\delta z + B\delta v \quad (3)$$

where  $\delta z$  is the component of the missile displacement error at cut-off along a certain displacement critical direction;

$\delta v$  is the component of the missile velocity error at cut-off along a certain velocity critical direction;

and  $A, B$  are constants depending on the trajectory.

The constants  $A, B$  are different from those in the preceding section but no confusion need arise.

The critical directions depend on the trajectory to be followed. For the particular trajectory on which the climb angle at cut-off is optimized to give maximum range over the orbital phase of the trajectory, the velocity critical direction lies along the desired direction of the velocity so that effectively  $\delta v$  is the error in speed (the scalar magnitude of the velocity). This ceases to be true on other types of ballistic trajectory.

Take as examples of errors to be corrected near cut-off

$$\delta z = 5 \text{ n. miles}$$

$$\delta v = 50 \text{ ft/sec},$$

similar to the errors in azimuth in the preceding section.

# SECRET - DISCREET

Technical Memorandum No. GW 263

Consider the same typical trajectory such that the missile travels 2500 n. miles from a cut-off 100 n. miles high, with the climb angle so chosen as to require least speed at cut-off.

Then

$$A = 0.4113 \times 10^{-3} \text{ n. miles per foot}$$

$$B = 0.3715 \text{ n. miles per ft/sec.}$$

The displacement critical direction is at 67.135 degrees to the horizontal and the velocity critical direction is at 33.50 degrees to the horizontal.

Equation (3) shows that the error to be corrected amounts to

$$E_r = 31.08 \text{ n. miles.}$$

3.2 Suppose it is required to correct the error in range by means of an acceleration  $f$  acting for a time  $\tau$ . Let the acceleration act at an angle  $\alpha$  to the horizontal where  $\alpha$  remains to be chosen.

At the end of the period of acceleration, the component of the displacement error along the critical direction is

$$\delta z + \frac{1}{2} f \tau^2 \cos (\alpha - 67^\circ 8')$$

and the component of the velocity error along the critical direction is

$$\delta v + f \tau \cos (\alpha - 33^\circ 30').$$

Thus at the end of the period of control acceleration, the range component of the impact error is

$$A \left\{ \delta z + \frac{1}{2} f \tau^2 \cos (\alpha - 67^\circ 8') \right\} + B \left\{ \delta v + f \tau \cos (\alpha - 33^\circ 30') \right\}$$

assuming that the constants  $A$  and  $B$  will remain nearly constant. As before, this is true provided the control duration  $\tau$  is short compared with the total time of flight over the trajectory. If the final error at impact is zero, then

$$f = - \frac{A \delta z + B \delta v}{\frac{1}{2} A \tau^2 (\cos \alpha - 67^\circ 8') + B \tau \cos (\alpha - 33^\circ 30')}.$$

Using equation (3), this may be written as

$$- \frac{E_r}{f} = \frac{1}{2} A \tau^2 \cos (\alpha - 67^\circ 8') + B \tau \cos (\alpha - 33^\circ 30') \quad (4)$$

which may be expressed in words as the range error at the target corrected by unit acceleration during the vernier control. Expression (4) is a function of  $\alpha$ , the direction of the correcting acceleration  $f$ . It is obviously most economical to choose the inclination  $\alpha$  so that the expression (4) is a maximum. This occurs when



**SECRET - DISCREET**

Technical Memorandum No. GW 263

$$\tan \alpha = \frac{\frac{1}{2}A\tau^2 \sin 67^\circ 8' + B\tau \sin 33^\circ 30'}{\frac{1}{2}A\tau^2 \cos 67^\circ 8' + B\tau \cos 33^\circ 30'} \quad (5)$$

Consider a vernier duration  $\tau = 20$  seconds. Then the value of the angle  $\alpha$  given by equation (5) is  $33^\circ 51'$ , which scarcely differs from the inclination of the velocity critical direction.

With this value of  $\alpha$ , the two components on the right hand side of expression (4) are

$$\frac{1}{2}A\tau^2 \cos (\alpha - 67^\circ 8') = 0.06877 \text{ n. miles per ft/sec}^2$$

$$B\tau \cos (\alpha - 33^\circ 30') = 7.43 \text{ n. miles per ft/sec}^2$$

From the relative sizes of these two terms in expression (4) the effect of a change of speed is over one hundred times the effect of the corresponding change in position. Again, the conclusion is that guidance in range should aim at achieving a certain critical component of the missile velocity (loosely equivalent to the speed) the specified component being a slowly changing function of climb angle and position.

3.3 It is worthy of note that the error in range is very insensitive to the climb angle at out-off. This is primarily due to the method of optimizing the ballistic trajectory proposed in reference 2. However, since the orbital part of the trajectory comprises most of the flight of the missile, the climb angle at out-off suggested by reference 2 is not far from the value which gives maximum range from launch to impact. Adjustment of the range error at impact by means of the climb angle must be a wasteful process requiring perhaps a hundred times as much acceleration as the adjustment of speed.

In order to correct the typical error in range  $E_r$  quoted above, the missile will require an acceleration of  $4.1 \text{ ft/sec}^2$  for 20 seconds.

#### 4 Reference velocity

4.1 The previous two sections have shown that during the vernier stage of guidance and control, it is profitable to regard the guidance in terms of the velocity with which the missile is moving. Corrections to the missile path are most readily communicated as small changes to the velocity already attained. Out of this method of thinking there arises the concept of a reference velocity, defined as the velocity which the missile should possess if it is to hit the target, towards which the actual missile velocity should be steered. The reference velocity is a slowly varying function of position, and of the climb angle which may be arbitrarily disposed. Guidance in azimuth may be attained by comparing the desired and attained values of the lateral velocity components and accelerating the missile to null the difference. Guidance in range could be achieved by comparing the reference speed with the speed attained and accelerating the missile to null the difference.

4.2 Such a method of guidance in azimuth appears quite feasible. However, for guidance in range, it appears sensible to arrange that the missile needs only to accelerate and never to decelerate. Such an arrangement represents an economy in fuel since there is no need to accelerate beyond the reference speed. Furthermore, there is no need to mount a motor which can thrust in a direction to oppose the velocity. The price paid for the provision of thrust only in the direction of motion is that the operation of controlling in range is a single attempt which must be sufficiently accurate at the first shot.

# SECRET - DISCREET

Technical Memorandum No. GW 263

There is no possibility of avoiding an error if the thrust is not cut-off at the correct instant since relighting the motor is impracticable and no motor is installed for providing backward thrust.

4.3 It appears desirable to compute continuously (strictly at frequent intervals) all three components of the reference velocity so that a frequent comparison of cut-off velocity and actual velocity may be made. At the final vernier cut-off, it is necessary to control all three components of the missile velocity to be adequately close to the corresponding components of the reference velocity. Since the moment of cut-off is governed by the coincidence of the critical components, the other two components of the missile velocity should be already adequately close to the reference components.

Hence there arises a distinction between guidance in range and in azimuth. The guidance in range consists of a single act when a signal is sent to cut off the motor. Guidance in attitude which governs primarily the azimuth error (see section 3.3 about effect of climb angle on range error) will be prolonged and should aim to be complete by the earliest moment at which cut-off is likely to occur. The term 'attitude' is used in a vector sense to denote the direction of the missile longitudinal axis in both vertical and horizontal planes. By the term "completion" is meant that the control in azimuth has reduced and maintained the difference between the lateral component of the reference velocity and the actual velocity below a level corresponding to tolerable error at impact. If the azimuth control were prolonged beyond the cut-off point of the longitudinal motor, it would be essential that the azimuth motors be at right angles to the velocity of the missile in order that no further error in range be caused by subsequent correction to the azimuth velocity component. Such a stipulation would be difficult to satisfy, and furthermore, the azimuth motors would be separate units. It is hoped that adequate attitude control may be achieved by sideways deflection of a motor which thrusts chiefly longitudinally.

4.4 The size of thrust during the vernier stage of flight is dictated by the ability to cut-off the speed and control the attitude of the missile within tolerances which correspond to the required accuracy in impact position. Reference 4 quotes that a North American rocket motor with 240 K lb thrust can be cut-off with an effective scatter in time of  $\pm 0.04$  second, a figure which may be improved upon, particularly if some attention is directed to achieving reproducible cut-off. Assuming the same scatter may be applied to the vernier motor which will have smaller thrust, the acceleration during the vernier stage may be no more than  $25 \text{ ft/sec}^2$  ( $0.78g$ ) in order that the speed after cut-off shall be correct to  $\pm 1 \text{ ft/sec}$ . For the Atlas missile, Convair proposals were for a vernier acceleration of  $0.25g$ .

The length of the vernier stage would be ideally as short as possible. In a guidance scheme employing radar, the missile is rapidly becoming more remote from the ground station so that measurements are progressively more difficult, and probably less accurate, as time proceeds. Similar considerations apply to inertia guidance equipment where some part of the error will increase with time. Thus it is desirable that the errors to be corrected at the start of the vernier stage shall be as small as possible. This means that guidance during the main boost, and the cutting off of the main motor should be as accurate as possible, suggesting that whatever method of guidance is used during the vernier control stage should also be employed during the later stage of the main boost phase.

If the final acceleration during boost is  $20g$  at the time of main motor cut-off, and the scatter in the effective cut-off instant is  $0.04$  second, the scatter in velocity after cut-off is likely to be  $\pm 26 \text{ ft/sec}$ . Thus if the main motor cut-off is timed to be effective on average at a speed  $50 \text{ ft/sec}$  short of the reference speed, the missile will rarely exceed the required

# SECRET - DISCREET

Technical Memorandum No. GW 263

cut-off speed as a result of uncertainty in the impulse arising during extinction of the main motor. If the vernier acceleration were chosen as  $10 \text{ ft/sec}^2$ , the vernier stage would last on an average 5 seconds.

However, it will be necessary to ensure that the vernier stage lasts for at least some time in order that guidance and control functions may have time to settle. Any lateral velocity at the instant of main motor cut-off must be reduced to a tolerable level. In the Convair scheme for the guidance of Atlas, the smoothing time constant used in deriving missile velocity was expected to be of the order of 10 seconds, and so the vernier stage was arranged to last for at least 20 seconds.

4.5 It is apparent that there may be a need for a reference velocity which is defined over a whole range of positions covering the vernier phase and also the later stages of the boost phase.

The first step in obtaining such a reference velocity is to derive a general relation for the velocity required to pass through a specified target. This may be deduced as follows, under the simplifying assumptions of a spherical non-rotating earth.

Take coordinates  $(r, \phi)$  to be polar coordinates with respect to the centre of the earth as origin and initial line passing through the target. The angle  $\phi$  is measured positively in the sense opposing the direction of flight of the missile.

Reference 2, equation (4), shows that the condition that an orbit starting from a point  $(r_1, \phi_1)$  passes through a target  $(r_2, 0)$  is

$$\frac{1}{r_2} = \frac{1+q_1^2}{p_1 r_1} (1 - \cos \phi_1) + (\cos \phi_1 - q_1 \sin \phi_1) \frac{1}{r_1} \quad (6)$$

where  $p_1, q_1$  govern the velocity at cut-off through the relations

$$p_1 = \frac{r_1 v_1^2}{g R^2} \quad (7)$$

$$\text{and } q_1 = \tan \theta_1. \quad (8)$$

At cut-off, the velocity is assumed to be of magnitude  $v_1$  inclined at an angle  $\theta_1$  to the local horizontal. The radius of the earth is taken as  $R$ , and  $g$  the acceleration due to gravity at the earth surface.

From the analytical geometry of an ellipse, it may be deduced from equation (6) that the length of the semi-latus-rectum of the orbit is

$$l = \frac{r_1 p_1}{1+q_1^2}. \quad (9)$$

$$\text{Write } e = \cos \frac{1}{2} \phi_1. \quad (10)$$

$$\text{It may be shown that } \cos \phi_1 = \frac{e^2 - 1}{e^2 + 1} \quad (11)$$

and  $\sin \theta_1 = \frac{2o}{o^2+1}$  (12)

Substituting equations (9), (11) and (12) into equation (6) gives

$$\frac{o^2+1}{r_2} = \frac{o^2+1-o^2+1}{\ell} + \frac{o^2-1-2q_1o}{r_1}$$

$$\therefore \frac{2r_1r_2}{\ell} = r_1(o^2+1) - r_2(o^2-1) + 2q_1or_2$$

Write  $h = r_1 - r_2$  (13)

where  $h$  is the height of the cut-off point in excess of the target height.

Then  $\frac{2r_1r_2}{\ell} = h(o^2+1) + 2r_2(1+q_1o)$  (14)

Substituting for  $\ell$  from equation (9) and for  $p_1$  from equation (7) results in the following equations:

$$v_1^2 = \frac{gR^2 p_1}{r_1} = \frac{2gR^2 r_2(1+q_1^2)}{r_1 \{h(o^2+1) + 2r_2(1+q_1o)\}} \quad (15)$$

4.6 Equation (15) defines the speed  $v_1$  required at a cut-off point distance  $r_1$  from the centre of the earth in order to reach a target distance  $r_2$  from the centre of the earth at a range from cut-off such that the trajectory subtends an angle  $\theta_1$  at the centre of the earth. In equation (15), it is necessary to specify the climb angle at cut-off  $\theta_1$  (governed by  $q_1 = \tan \theta_1$ ). The variables  $o$  and  $h$  are defined in equations (10) and (13). It is apparent that an infinity of vacuum trajectories may be found passing through two specified points, at cut-off and the target. Generally, it is necessary to specify in addition some extra criterion in order to define uniquely the velocity required at cut-off (reference velocity). This may be achieved for example by making the climb angle some specified function of position. The reference velocity then becomes a definite function of position, with the magnitude  $v_1$  governed by equation (15).

## 5 Some definitions of reference velocity

5.1 Some six simple methods of defining the reference velocity are considered below. The first definition is perhaps the most obvious and comprises the velocity which has the least magnitude of all those which cause the missile to pass through the target. The reference velocity defined in this way is the optimum one suggested in reference 2 and is most simply defined by the formulae quoted there. The speed is shown in equations (26) and (29) of reference 2 to be  $v_1$  where

$$v_1^2 = \frac{2gR^2(d-h)}{r_1(r_1+r_2+d)} \quad (16)$$

where  $d$  is the straight line distance from cut-off to the target defined by

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos \theta_1. \quad (17)$$

This equation (17) may be written in a slightly more convenient form by use of equation (13) as

$$d^2 = h^2 + 2r_1r_2 (1 - \cos \theta_1). \quad (17a)$$

The climb angle required on such an optimum trajectory is  $\theta_1$  where equation (33) of reference 2 shows

$$\cot 2\theta_1 = \left( \frac{r_1}{r_2} \right) \operatorname{cosec} \theta_1 - \cot \theta_1. \quad (18)$$

5.2 Another possible definition of the reference velocity arises from the work of reference 1, where it is assumed that the missile flies on a ground optimized trajectory. This is defined as the vacuum ballistic trajectory passing through the cut-off point and the target which covers the greatest range between the two points in which it intersects the earth surface. Inclusion of an "imaginary" part of the trajectory before the cut-off point makes some allowance for the necessity to choose a trajectory which is economical between launch and target rather than between cut-off and target as in reference 2. As shown in reference 1, equation (27), the condition which specifies a ground optimized trajectory is

$$p_2 = 1 - q_2^2. \quad (19)$$

By analogy with equation (15), interchanging impact and out-off,

$$p_2 = \frac{2r_1(1+q_2^2)}{-h(\sigma^2+1) + 2r_1(1-\alpha q_2)}. \quad (20)$$

Thus, substituting equation (20) into condition (19), gives

$$\begin{aligned} 2r_1(1-\alpha q_2) - h(\sigma^2+1) &= 2r_1 \left( \frac{1+q_2^2}{1-q_2^2} \right) \\ 2r_1\alpha q_2 + 2r_1 \left( \frac{1+q_2^2}{1-q_2^2} - 1 \right) + h(\sigma^2+1) &= 0 \\ \text{i.e., } \frac{q_2^2}{1-q_2^2} + \frac{1}{2}\alpha q_2 + \frac{(1+\sigma^2)h}{4r_1} &= 0. \end{aligned} \quad (21)$$

Equation (21) represents a cubic in  $q_2$  which may be solved to determine the trajectory required. Equation (21) as written is in a form suitable for rapid solution by numerical approximation.

It has been shown in reference 3, equation (7) that

$$\cot \frac{1}{2} \theta_1 = - \frac{r_1 q_2 + r_2 q_1}{r_1 - r_2}.$$

Substituting from equations (10) and (13) gives

$$r_1 q_2 + r_2 q_1 + ch = 0. \quad (22)$$

Once a value of  $q_2$  has been determined from equation (21), the use of equation (22) leads to the corresponding value of  $q_1$  and so the climb angle at out-off by equation (8). When the value of  $q_1$  has been determined, the speed at out-off  $v_1$  may be calculated from equation (15).

5.3 A third definition of the reference velocity arises from maintaining the climb angle at out-off a constant over the whole family of trajectories. The direction of the reference velocity is thus specified from the outset and the magnitude  $v_1$  may be calculated readily from equation (15). As an example below, the constant value used is approximately equal to the optimum climb angle at the standard out-off position; optimum being used in the sense of reference 2 as the climb angle which requires least speed to reach the target from the given out-off point.

5.4 A slight modification of maintaining the climb angle constant would be to maintain the out-off velocity parallel to a fixed direction in space. Such a definition of climb angle might be more convenient than maintaining the climb angle constant, since the climb angle may be determined only when the plan position of the missile and hence the local vertical is known. The method of calculation is as above in section 5.3.

5.5 A more sophisticated suggestion originating with Convair, USA, is that the locus rectum of the family of trajectories should be a specified constant. This may be expressed in an equivalent form by requiring that the angular momentum about the earth centre per unit mass of the missile is the same for all the trajectories. As an example below, the constant value used is the rounded value of that on the optimum trajectories of section 5.1 evaluated at the standard out-off position.

From equation (14) above, it can be seen that an equation exists relating the climb angle parameter  $q_1$  with the out-off position

$$q_1 c = \frac{r_1}{\ell} - \frac{h}{2r_2} (c^2 + 1) - 1. \quad (23)$$

Equation (23) defines the climb angle  $\theta_1$  uniquely in terms of the out-off position through the variables  $r_1$  and  $c$ . When the value of  $q_1$  has been calculated, the corresponding reference speed  $v_1$  may be found from equations (7) and (9)

$$v_1^2 = \frac{g R^2 p_1}{r_1} = \frac{g R^2 \ell (1 + q_1^2)}{r_1^2}. \quad (24)$$

5.6 A final definition of reference velocity arises by analogy with the previous definition. The other principal length associated with an ellipse

is its major axis and it is possible to imagine a scheme in which the length of the major axis of all the elliptical orbits is specified. Equation (21) in reference 1 shows that the length of the semi-major axis is

$$a = \frac{r_1}{2-p_1} \quad (25)$$

Thus it follows from equation (7) that

$$v_1^2 = \frac{g R^2 p_1}{r_1} = 2g R^2 \left( \frac{1}{r_1} - \frac{1}{2a} \right) \quad (26)$$

Equation (26) shows that the reference speed required by this relation is purely a function of height and does not depend on the distance to the target. This might possess some advantage in simplicity. However, there is an associated disadvantage in freeing the reference speed from dependence on the plan position of the cut-off point. In order that the missile may reach the target, it is necessary that the speed specified by equation (26) shall exceed the minimum speed specified by equation (16) of section 5.1. This means that the value chosen for the length of the major axis (2a) must be adequate for all expected positions at which the reference velocity is to be computed. Otherwise the corresponding limb angle will be imaginary. It may be shown (reference 2) that the appropriate value for the major axis is half the maximum value of the perimeter of the triangle formed by the cut-off point, the target and the centre of the earth.

The corresponding climb angle may be deduced as follows. Substituting equation (9) in equation (14) gives

$$\frac{2r_2(1+q_1^2)}{p_1} = (c^2+1)h + 2r_2(1+q_1c)$$

$$\text{i.e.} \quad q_1^2 + 1 = p_1 \left\{ \frac{h(c^2+1)}{2r_2} + 1 + q_1c \right\}$$

$$\therefore (q_1 - \frac{1}{2}cp_1)^2 = \frac{1}{2}p_1(c^2+1) \left( \frac{1}{2}p_1 + \frac{r_1}{r_2} - 1 \right) - (1 - \frac{1}{2}p_1)^2$$

Substitute for  $p_1$  from equation (25)

$$(q_1 - \frac{1}{2}cp_1)^2 = r_1^2(c^2+1) \left( \frac{1}{r_1} - \frac{1}{2a} \right) \left( \frac{1}{r_2} - \frac{1}{2a} \right) - \frac{r_1^2}{4a^2}$$

Thus

$$q_1 = \alpha_1 \left( \frac{1}{r_1} - \frac{1}{2a} \right) \pm r_1 \left\{ (c^2+1) \left( \frac{1}{r_1} - \frac{1}{2a} \right) \left( \frac{1}{r_2} - \frac{1}{2a} \right) - \left( \frac{1}{2a} \right)^2 \right\}^{\frac{1}{2}} \quad (27)$$

Either of these values of the climb angle will cause the missile to pass through the target. For the example below, the lower climb angle has been quoted as more appropriate. It may be shown that the values of  $q_1$  given by expression (27) are real so long as the speed corresponding to the value of (26) exceeds the minimum speed required to reach the target.

5.7 The above examples of reference velocities have been computed below for a selection of five positions. The standard trajectory is taken to be one on which the missile travels 2500 n. miles to a target on the ground from a cut-off point 100 n. miles high. About this cut-off point which has been designated (2500, 100) four other points have been considered differing by  $\pm 50$  n. miles in height and  $\pm 100$  n. miles in ground range: viz.

(2400, 100) (2500, 150) (2600, 100) (2500, 50).

It will be observed that the first of these coordinates represents the ground range travelled from cut-off to impact and the second the height of cut-off, both measured in nautical miles.

The six values of reference velocities are compared in the table below for each of the five positions enumerated. The first table lists values of the reference speeds and the second table the corresponding climb angles.

Table of reference speeds (ft/sec)

Position	2500, 100	2400, 100	2500, 150	2600, 100	2500, 50
Least speed	18095.74	17821.06	17789.87	18359.54	18410.18
Ground optimum	18109.45	17835.41	17822.70	18372.68	18413.38
$\theta_1 = 33\frac{1}{2}^\circ$	18095.74	17821.61	17790.92	18360.07	18411.35
Const. inclination	18095.74	17827.66	17790.92	18368.12	18411.35
$\ell = 1239.418$	18095.74	17832.19	17790.13	18370.36	18410.48
$2a = 4776.287$	18359.54	18359.54	17859.37	18359.54	18860.21

Table of reference climb angles  
(inclination in degrees to local horizontal)

Position	2500, 100	2400, 100	2500, 150	2600, 100	2500, 50
Least speed	33°30'16"	33°52'20"	32°58'40"	33°8'	34°2'24"
Ground optimum	31°38'44"	31°57'49"	30°3'37"	31°19'7"	33°9'4"
$\theta_1 = 33\frac{1}{2}^\circ$	33°30'	33°30'	33°30'	33°30'	33°30'
Const. inclination	33°30'	35°10'	33°30'	31°40'	33°30'
$\ell = 1239.418$	33°30'	32°11'54"	33°14'21"	34°46'22"	33°45'33"
$2a = 4776.287$	25°24'28"	22°21'29"	28°44'15"	33°8'	23°40'24"

5.8 Inspection of the two tables above shows that maintaining the climb angle constant demands hardly more than 1 ft/sec greater speed than the least. The climb angles may differ by up to half a degree over the region of space considered, but the reference speeds hardly differ significantly.



Maintaining the inclination of the reference velocity constant in space causes a slightly greater demand for speed (almost another 10 ft/sec) and is associated with a variation of rather more than a degree about the climb angle for least speed. A similarly close approximation to the least speed is provided by the Convair proposal of constant *latus rectum*. The speeds demanded approximate closely to those for a constant velocity direction in space (up to 10 ft/sec greater than the least) and the climb angles again deviate about a degree from the climb angle for least speed but in the opposite sense to the deviations of the constant inclination.

The reference speeds demanded by a ground optimized trajectory are as much as 30 ft/sec greater than the least, and the climb angles up to 3 degrees lower. The climb angles are always lower on the ground optimized trajectory than the climb angles for least speed, but as the height decreases to zero, the two values approach equality.

The only definition of reference speed considered here which leads to speeds greatly exceeding the least is that for which the major axis of the trajectories is specified. This arises because the value of the major axis must be chosen large enough to ensure at least the lowest speed is demanded at all points where the reference velocity is to be evaluated. For the five positions considered in the table, the major axis was determined by the point (2600, 100) at which the reference velocity was chosen equal to that for the lowest speed. However, at the point (2400, 100) the resulting speed for a constant major axis is over 500 ft/sec greater than the lowest speed, which would demand a large sacrifice in performance. Unless the variation of cut-off position can be more closely circumscribed, the constant major axis determination of reference velocity does not appear profitable.

## 6 Linear and quadratic fits to reference velocity

6.1 For simplicity in computing the reference velocity (e.g. in an airborne computer) it may be desired to approximate to the reference speed and climb angle. A numerical investigation is described below which determines the range of positions over which linear and quadratic approximations will hold.

The investigation has been carried out in terms of two coordinates of position, viz. the ground range from launch and the height above the earth surface. No regard has been paid to the dependence on the azimuth coordinate. The ground range and height are the two position coordinates which arise most naturally in the analysis about a spherical earth but it must be noted that they constitute curvilinear coordinates. Because of this curvature, transformation to other coordinate systems is not simple. This means that the results are not necessarily applicable to a computer fed by an inertia navigator which measures displacements and velocities along certain fixed directions.

The definition of reference velocity chosen for much of the work is that given by a constant climb angle. This definition is simpler for numerical work than the others, which may recommend it also for guidance purposes.

Since the climb angle is chosen constant at all points where the reference velocity is computed, only the reference speed varies. The constant climb angle chosen was  $33\frac{1}{2}$  degrees to the local horizontal which is nearly that requiring least reference speed at the standard cut-off position. The standard cut-off position is the same as that above in section 5 with cut-off 100 n.miles high and impact at a ground range from cut-off of 2500 n. miles. The reference speed at the standard cut-off position will be referred to as the standard reference speed.

6.2 Suppose the reference speed is approximated by a linear relationship. Consider the approximation evaluated at a speed  $(1+\epsilon)$  times the standard

reference speed, where  $\epsilon$  is a small fraction. The next term in the Taylor series in which the reference speed is expanded about the standard cut-off point shows that errors between a linear approximation and the true reference speed are of order  $\epsilon^2$ . If an error of  $\frac{1}{2}$  ft/sec is tolerable in the approximating, and the standard value of the reference speed is about 20 K ft/sec, then to better than an order of magnitude

$$\epsilon^2 = \frac{1}{2 \times 20,000}$$

i.e.  $\epsilon = \frac{1}{200}.$

Hence it follows that a linear approximation may be expected to hold adequately for reference speeds within  $\pm 100$  ft/sec of the standard reference speed. This is confirmed below.

Similarly with the quadratic approximation, errors between it and the accurate value have the order  $\epsilon^3$ . Thus to better than an order of magnitude

$$\epsilon = \frac{1}{34}.$$

Thus a quadratic approximation may be expected to hold adequately over a spread of  $\pm 600$  ft/sec in the reference speeds.

6.3 Appendix I contains a derivation of the first and second order differential coefficients of the reference speed, defined by a fixed climb angle. The reference speed is a function of two variables, the ground range and height of cut-off. Variation of the reference speed with the climb angle need not be considered since the climb angle is constant. The equations are numbered in the same sequence as those above in the main text.

The expressions derived for the two first order differentials and the three second order differentials are evaluated at the standard cut-off position. The results are in agreement with those published in Reference 3.

6.4 Working correct to seven figures, the following table of reference speeds may be computed for the constant climb angle  $\theta_1 = 33\frac{1}{2}$  degrees.

Table of reference speeds (ft/sec): constant climb angle

		Ground range to impact (n. miles)		
		2470	2500	2530
Out-off	85	18108.44	18189.26	18269.20
height	100	18014.56	18095.74	18176.03
(n. miles)	115	17921.66	18003.18	18083.82

Working equally accurately using the first order coefficients derived in Appendix I, a similar table may be compiled of the linear approximation to the reference speed. The following table shows the differences between the linear approximation and the accurate values at the nine points shown above.

Table of differences between linear approximation  
and accurate reference speed (ft/sec)

		Ground range to impact (n. miles)		
		2470	2500	2530
Cut-off	85	-0.41	-0.49	+0.30
height	100	+0.44	0	+0.44
(n. miles)	115	+0.31	-0.48	-0.39

This table confirms that a linear approximation to the reference speed is likely to hold to within  $\frac{1}{2}$  ft/sec for reference speeds differing by 100 ft/sec from the standard cut-off position.

By computing a considerable number of differences in the manner described above, the diagram in Fig. 1 has been compiled. This shows the contour of  $\frac{1}{2}$  ft/sec difference between the linear approximation and the accurate reference speed. The contour may be treated in the same way as a polar diagram in that the corresponding contour for say 1 ft/sec may be obtained by drawing in the curve which is at all points  $(2)^2$  times as far away from the origin as the curve shown. This is because (for small errors) the error in the approximation varies as the square of the tolerance permitted on the variation in position (see section 6.2).

Figure 1 shows that the linear approximation will be adequately close over a range of cut-off positions within  $\pm 15$  n. miles of the standard cut-off height and  $\pm 30$  n. miles of the standard cut-off ground range.

6.5 In a similar way, the coverage of the quadratic approximation may be determined. The following table lists the reference speeds computed for a constant climb angle of  $33\frac{1}{2}$  degrees. The positions are more widely spread than in the table of section 6.4.

Table of reference speeds (ft/sec): constant climb angle

		Ground range to impact (n. miles)		
		2400	2500	2600
Cut-off	50	18141.54	18411.35	18671.71
height	100	17821.61	18095.74	18360.07
(n. miles)	150	17512.97	17790.92	18058.79

Using all the five first and second order coefficients derived in Appendix I, the corresponding values of the quadratic approximation may be computed. The following table shows differences between the quadratic approximation and the accurate values quoted in the table above.

Table of differences between quadratic approximation  
and accurate reference speed (ft/sec)

		Ground range to impact (n. miles)		
		2400	2500	2600
Cut-off	50	-0.39	-0.10	-0.13
height	100	+0.12	0	-0.12
(n. miles)	150	+0.13	0.09	0.34

By computing the differences for several more points, the diagram in Fig.2 has been plotted. This shows the contour corresponding to  $\frac{1}{2}$  ft/sec error in the quadratic approximation to the reference speed. As in Fig.1 the contour may be scaled in or out since the errors in the approximation vary roughly as the cube of the distance from the standard position at the origin. Thus the contour for any other error value may be obtained by magnifying or diminishing the scale of Fig.2.

The results expressed by Fig.2 are more difficult to summarize than for Fig.1. It appears reasonable to say that the quadratic approximation will hold adequately over a range of cut-off positions within  $\pm 75$  n. miles of the standard position measured normal to the trajectory and  $\pm 150$  n. miles measured along the trajectory.

6.6 Some work was also completed on a definition of reference velocity such that the climb angle is chosen to make the reference speed least. Appendix II contains a derivation of the first and second order differential coefficients of the reference speed, and the first differentials of the climb angle. The expressions are evaluated at the standard cut-off position.

The following table of reference speeds has been computed for the least reference speed at the same points as in section 6.5.

Table of reference speeds (ft/sec): least reference speed

		Ground range to impact (n. miles)		
		2400	2500	2600
Cut-off	50	18138.01	18410.16	18671.62
height	100	17821.06	18095.74	18359.54
(n. miles)	150	17512.85	17789.87	18055.84

Using the differential coefficients evaluated in Appendix II, the corresponding values of the quadratic approximation may be calculated. The following table shows differences between the quadratic approximation and the accurate values of the reference speed for the nine positions shown in the table above.

# SECRET - DISCREET

Technical Memorandum No. GW 263

Table of differences between quadratic approximation  
and accurate reference speed (ft/sec)

		Ground range to impact (n. miles)		
		2400	2500	2600
Cut-off	50	-0.11	-0.05	-0.16
height	100	0.12	0	-0.11
(n. miles)	150	0.16	0.04	0.10

Comparison with the corresponding table in the preceding section 6.5 shows that the quadratic approximation to the least reference speed is rather closer than the quadratic approximation to the reference speed for constant climb angle. Thus the quadratic approximation to the least reference speed may be expected to hold over a somewhat larger area than that described at the end of section 6.5.

Since the climb angle varies when the reference velocity is defined as that requiring least speed, it is necessary also to approximate to the variation of the climb angle. The following table shows accurate values of the climb angle computed for the same positions as the reference speeds above.

Table of climb angles for least reference speed  
(degrees to local horizontal)

		Ground range to impact (n. miles)		
		2400	2500	2600
Cut-off	50	34°25'55"	34°2'24"	33°38'46"
height	100	33°52'20"	33°30'17"	33°8'10"
(n. miles)	150	33°19'17"	32°58'40"	32°37'43"

Appendix II contains the analysis leading to values of the first derivatives of the climb angle with respect to height and ground range. Thus the corresponding values of the linear approximation to the climb angle may be calculated. The following table shows differences between the linear approximation to the climb angle and the accurate values quoted in the table above.

Table of differences between linear approximation  
and accurate climb angle (seconds of arc)

		Ground range to impact (n. miles)		
		2400	2500	2600
Cut-off	50	-96	-15	73
height	100	7	0	7
(n. miles)	150	78	-15	-88

It may be shown (e.g. Reference 1) that for climb angles differing by up to 10 minutes of arc from the climb angle for least reference speed, there is insignificant effect on the reference speed. Thus it appears from the above table that a linear approximation to the climb angle would be adequate over the range of positions described at the end of section 6.5.

6.7 The size of each of the linear terms in the approximation to the reference speed rises to about 300 ft/sec at the edge of the region covered by the tables in sections 6.5 and 6.6. If error in the linear coefficient is to contribute no more than  $\frac{1}{2}$  ft/sec error to the value of the reference speed, the coefficient must be accurate to within two parts in a thousand.

The size of the quadratic terms in the approximation to the reference speed rises to about 5 ft/sec at the edge of the region covered by the tables. Thus if the error in the quadratic coefficient is to contribute no more than  $\frac{1}{2}$  ft/sec error to the reference speed, the coefficient must be accurate to about 10%.

Over the region described at the end of section 6.5, the relative accuracies required in the coefficients of the expansion of the reference speed are as follows:

Constant term:	2 parts in 100,000
Linear coefficient:	1 part in 1,000
Quadratic coefficient:	3 parts in 100.

The last two accuracies are probably within the capability of an analogue computing device. Thus the reference speed computer may be based on analogue circuits provided the origin is shifted to the standard value of the reference speed.

## 7 Some possible guidance principles

7.1 As a result of the work above, it seems clear that firm proposals may be made for guidance during the vernier stage of thrust. These are outlined below in the conclusions. Concerning guidance during the period of main motor thrust, the work above suggests two broad principles. However, the velocity of the missile is a three dimensional vector, so that there are three disposable components. The employment of only two guidance principles would leave guidance in the third dimension arbitrary. Guidance in range is achieved by the choice of cut-off speed. Guidance to remove the line error at impact is by controlling the left/right component of the missile velocity. Guidance in the up/down direction, which governs the climb angle and attitude angle, has not been defined so far.

7.2 The choice of the climb path affects considerably the maximum range which the missile can travel for a given motor performance. Reference 5 has shown that a close approximation to the best performance is given by a climb at constant attitude (after a turnover through the atmosphere from a vertical launch direction). The value of the optimum attitude angle is a function of the range to be traversed and of the motor performance.

7.3 A climb at constant attitude could be achieved by guidance from an auto-pilot in which the free gyro is sufficiently wander-free to maintain its direction throughout the boost period. If such a guidance scheme were adopted, the reference velocity could be computed as suggested for the vernier phase from measurements of the position and climb angle of the missile. Guidance in azimuth could be either by comparing left/right components of the actual and reference velocities or by constant pre-set attitude.

Work is needed on this guidance scheme to investigate whether a satisfactory system can be developed. On the one hand the control system must not be so heavy or demand so much power as to cause a considerable loss in performance. On the other hand, the control must not be so slack as to allow wide dispersion which would again cause loss of performance through mis-direction of the rocket thrust or by requiring a heavier guidance computer. Dispersion at main motor cut-off also determines the size of the vernier control equipment.

7.4 Alternative guidance schemes may be devised in which the guidance is not preset as in the scheme above. For example, the following principle appears workable although no investigation has been carried out. A preferred attitude is defined which might be, for example, the direction along which the thrust of the missile would be directed in the preset scheme above. The missile is controlled in heading so as to make the up/down and left/right velocity components approach the corresponding components of the reference velocity (defined with respect to the preferred attitude). If the reference velocity is chosen suitably, the components normal to the preferred attitude will change little along the standard trajectory. Possibly one of the definitions of reference velocity discussed in the preceding section will suffice for this purpose. Under these conditions, if the missile suffers small dispersion, the missile attitude and thrust direction will soon settle along the preferred attitude. If, however, the missile suffers a wide dispersion, the components of the reference velocity will change slowly and cause the missile to alter its heading, compensating to some extent for the dispersion.

This method of guidance appears more complicated in principle and might be more difficult to instrument. However, it does not require a gyroscope capable of preserving its attitude for the duration of the boost. There might also be less dispersion than with the preset method of 7.3.

7.5 Another guidance scheme (proposed by Convair) has been described as "pursuit homing in velocity space". The guidance computer is so constructed as to set up effectively two continuously varying vectors, one representing the reference velocity and the other the actual missile velocity. The guidance principle is to direct the heading of the missile along the vector difference, called the "velocity to be gained". If for purposes of illustration, the vector velocities are regarded as defining the positions of two points (target and missile) it will be appreciated that the guidance principle is akin to commanding the missile to perform a pursuit homing course towards the target.

This guidance scheme appears equally as flexible as the one described above in section 7.4 but suffers from one obvious defect. Towards the end of the boost phase when the actual velocity becomes nearly equal to the reference velocity, the guidance suffers from the well known failing of pursuit courses in that the missile heading is liable to change very rapidly.

## 8 Conclusions

8.1 The result of the arguments in chapters 2, 3 and 4 is that guidance during the vernier phase of thrust should aim at correcting the two velocity components in the critical and left/right directions. This suggests that the thrust of the vernier motor should be substantially along the critical direction with ability to deflect laterally for control of heading.

Guidance in the up/down plane may be chosen arbitrarily. Thus the attitude could be maintained constant by a free gyro preset before launch. Alternatively, as in the Convair proposals, the attitude could be maintained the same as at main motor cut-off by means of an auto-pilot gyro of shorter stability.

The reference velocity is computed as a function of the missile position and climb angle measured by the navigational equipment. Guidance in azimuth

aims at equating the left/right component of the missile velocity with the left/right component of the reference velocity, and maintaining the error adequately small. Guidance in range is achieved by cutting off the vernier motor when the missile critical velocity component reaches the corresponding critical component of the reference velocity.

8.2 The reference speed can be computed on the ground as a function of position by means of a relatively simple computer provided the coordinates are curvilinear ones, viz. ground range and height. If the dispersion of the cut-off position from the standard position does not exceed  $\pm 15$  n. miles in height and  $\pm 30$  n. miles in ground range, a linear approximation will give values within  $\frac{1}{2}$  ft/sec of the correct reference speed. A quadratic approximation lies within  $\frac{1}{2}$  ft/sec of the correct speed for dispersion of  $\pm 75$  n.miles normal to the trajectory and  $\pm 150$  n. miles along the trajectory.

8.3 Two guidance principles are suggested in sections 7.3 and 7.4 which appear worthy of investigation. Future investigation will take the form of devising the best control system and finding its effect on weight and performance of the missile.

#### Acknowledgement

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#### Glossary

Attitude is used to denote the direction in space of the missile longitudinal axis.

Suffices 1 and 2 are used to denote values of variables at cut-off and at impact.

- a length of semi major axis of elliptical orbit
- A factor relating error at impact to error in position at cut-off:  
see equations (1) - (5)
- $\alpha$  inclination to horizontal of vernier acceleration
- B factor relating error at impact to error in velocity at cut-off:  
see equations (1) - (5)
- c cut  $\frac{1}{2}g$
- d straight-line distance from cut-off to impact
- $E_L$  error in line of impact position relative to target
- $E_R$  error in range of impact " " " "
- f magnitude of vernier acceleration
- g acceleration due to gravity at earth surface:  $32 \text{ ft/sec}^2$
- h =  $x_1 - x_2$ : height of cut-off in excess of the impact point
- $\theta$  climb angle: inclination to local horizontal of missile velocity



# SECRET - DISCREET

Technical Memorandum No. GW 263

- $l$  length of semi latus rectum of elliptical orbit
- $\alpha_i$  angle subtended at the centre of the earth between the out-off point and the impact point
- $p$  parameter related to missile speed:  $= \frac{rv^2}{gR^2}$
- $q = \tan \theta$
- $r$  distance of missile from centre of the earth
- $R$  radius of earth 3437.75 n. miles
- $\tau$  duration of vernier acceleration
- $v$  speed of missile
- $x$  ground range of missile from launch
- $y$  displacement of missile normal to plane of desired trajectory
- $z$  displacement of missile along critical direction

## REFERENCES

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Attached:- Appendices I and II  
Drgs GW/P/6804, 6805

## Distribution

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APPENDIX I

First and Second Differentials of Reference Speed  
for Fixed Climb Angle

It is shown in the text, equation (15), that

$$v_1^2 = \frac{2g R^2 r_2(1+q_1^2)}{r_1 \{h(o^2+1) + 2r_2(1+oq_1)\}}.$$

Regard this as a definition of the reference speed  $v_1$  in terms of the cut-off position  $(r_1, \theta_2)$  of which the coordinates are measured with respect to the centre of the earth as origin and the direction of the target as initial line. The variables  $(r_1, \theta_2)$  govern the values of  $h$  and  $o$  through equations (10) and (13). The climb angle  $\theta_1$  is kept constant and so equation (8) shows that  $q_1$  is constant.

Differentiate logarithmically with respect to  $r_1$ ;

$$\frac{2\delta v_1}{v_1} + \frac{\delta r_1}{r_1} + \frac{(o^2+1)\delta r_1}{h(o^2+1) + 2r_2(1+oq_1)} = 0$$

using equation (13) which shows  $\delta h = \delta r_1$ .

Thus

$$\delta v_1 = -\frac{v_1 \delta r_1}{2r_1} \left\{ 1 + \frac{(o^2+1)r_1}{(o^2+1)h + 2r_2(1+oq_1)} \right\}. \quad (28)$$

Take for the standard out-off conditions a ground range to impact of 2500 n. miles at a climb angle of  $33\frac{1}{2}^\circ$  from a cut-off height of 100 n. miles.

Thus

$$\begin{aligned} q_1 &= \tan 33^\circ 30' = 0.66188556 \\ o &= \cot 20^\circ 50' = 2.627912 \\ h &= 100 \text{ n. miles.} \end{aligned}$$

Take the radius of the earth  $R = 3437.75$  n. miles and the acceleration due to gravity at the earth surface  $32 \text{ ft/sec}^2$ .

Equation (15) shows that

$$v_1 = 18095.74 \text{ ft/sec.}$$

Equation (13) shows that

$$r_1 = 3537.75 \text{ n. miles.}$$

Substituting the above values in expression (28) gives

$$\delta v_1 = -6.202416 \text{ ft/sec per n. mile in height.}$$

Differentiate equation (15) logarithmically with respect to  $\sigma$ ;

$$\frac{2\delta v_1}{v_1} = - \frac{(2h\sigma + 2r_2 q_1) \delta \sigma}{h(\sigma^2 + 1) + 2r_2(1 + \sigma q_1)}$$

$$\text{i.e.} \quad \delta v_1 = - \frac{v_1 \delta \sigma (h\sigma + r_2 q_1)}{h(\sigma^2 + 1) + 2r_2(1 + \sigma q_1)} \quad (29)$$

Now by equation (10)

$$\sigma = \cot \frac{1}{2} \Phi_1.$$

$$\text{Thus} \quad \delta \sigma = -\frac{1}{2}(\sigma^2 + 1) \delta \Phi_1 \quad (30)$$

Write the ground range from the launch point to cut-off as  $x_1$ . Then it follows that for fixed launch and impact positions,

$$\delta x_1 = -R \cdot \delta \Phi_1 \quad (31)$$

Substituting equation (31) into equation (30) gives

$$\delta \sigma = \left( \frac{\sigma^2 + 1}{2R} \right) \delta x_1 \quad (32)$$

Substituting equation (32) into equation (29) gives

$$\delta v_1 = - \frac{v_1 \delta x_1}{2R} \cdot \frac{(\sigma^2 + 1)(h\sigma + r_2 q_1)}{h(\sigma^2 + 1) + 2r_2(1 + \sigma q_1)} \quad (33)$$

Substituting standard values of the variables in expression (33) gives

$$\delta v_1 = -2.69113 \text{ ft/sec per n. mile ground range.}$$

Equation (28) may be written as

$$\frac{\partial v_1}{\partial r_1} = -\frac{v_1}{2} \left\{ \frac{1}{x_1} + \frac{(\sigma^2 + 1)}{(\sigma^2 + 1)h + 2r_2(1 + \sigma q_1)} \right\}.$$

Differentiating once more with respect to  $r_1$  gives

$$\begin{aligned} \frac{\partial^2 v_1}{\partial r_1^2} &= \frac{\partial v_1}{\partial r_1} \left( \frac{1}{v_1} \cdot \frac{\partial v_1}{\partial r_1} \right) + \frac{v_1}{2r_1^2} + \frac{v_1(\sigma^2 + 1)^2}{2\{(\sigma^2 + 1)h + 2r_2(1 + \sigma q_1)\}^2} \\ &= \frac{1}{v_1} \left( \frac{\partial v_1}{\partial r_1} \right)^2 + \frac{2}{v_1} \left[ \left( \frac{\partial v_1}{\partial r_1} \right)^2 - \frac{v_1^2(\sigma^2 + 1)}{2r_1\{(\sigma^2 + 1)h + 2r_2(1 + \sigma q_1)\}} \right] \end{aligned}$$

$$\text{i.e.} \quad \frac{\partial^2 v_1}{\partial x_1^2} = \frac{3}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 - \frac{v_1 (\sigma^2 + 1)}{r_1 \{ (\sigma^2 + 1)h + 2r_2(1 + \sigma q_1) \}} \quad (34)$$

Evaluated at the standard cut-off position, the above expression (34) is

$$\frac{\partial^2 v_1}{\partial x_1^2} = 0.004317 \text{ ft/sec per (n. mile)}^2.$$

Equation (33) may be written

$$\frac{\partial v_1}{\partial x_1} = -\frac{v_1}{2R} \cdot \frac{(\sigma^2 + 1)(h + r_2 q_1)}{h(\sigma^2 + 1) + 2r_2(1 + \sigma q_1)}.$$

Differentiating logarithmically with respect to ground range  $x_1$  gives

$$\begin{aligned} \frac{\partial^2 v_1}{\partial x_1^2} \bigg/ \frac{\partial v_1}{\partial x_1} &= \frac{\frac{\partial v_1}{\partial x_1}}{v_1} + \left( \frac{2\sigma}{\sigma^2 + 1} \right) \left( \frac{\partial \sigma}{\partial x_1} \right) + \left( \frac{h}{h + r_2 q_1} \right) \left( \frac{\partial \sigma}{\partial x_1} \right) \\ &\quad - \frac{2(h + r_2 q_1)}{h(\sigma^2 + 1) + 2r_2(1 + \sigma q_1)} \left( \frac{\partial \sigma}{\partial x_1} \right). \end{aligned}$$

Now equation (32) shows that

$$\frac{\partial \sigma}{\partial x_1} = \frac{\sigma^2 + 1}{2R}.$$

Thus

$$\begin{aligned} \frac{\partial^2 v_1}{\partial x_1^2} &= \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 + \left( \frac{\partial v_1}{\partial x_1} \right) \left[ \frac{\sigma}{R} + \frac{h(\sigma^2 + 1)}{2R(h + r_2 q_1)} \right. \\ &\quad \left. - \frac{(h + r_2 q_1)(\sigma^2 + 1)}{R \{ h(\sigma^2 + 1) + 2r_2(1 + \sigma q_1) \}} \right]. \end{aligned}$$

Using equation (33) to eliminate the last term gives

$$\frac{\partial^2 v_1}{\partial x_1^2} = \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 + \left( \frac{\partial v_1}{\partial x_1} \right) \left\{ \frac{\sigma}{R} + \frac{h(\sigma^2 + 1)}{2R(h + r_2 q_1)} \right\} + \frac{2}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2$$

$$\text{i.e.} \quad \frac{\partial^2 v_1}{\partial x_1^2} = \frac{3}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 + \left( \frac{\partial v_1}{\partial x_1} \right) \left\{ \frac{\sigma}{R} + \frac{h(\sigma^2 + 1)}{2R(h + r_2 q_1)} \right\}. \quad (35)$$

Evaluated at the standard cut-off position, the above expression (35) is

$$\frac{\partial^2 v_1}{\partial x_1^2} = -0.0009784 \text{ ft/sec per (n. mile)}^2.$$

Differentiating equation (33) logarithmically with respect to  $x_1$  gives

$$\frac{\partial^2 v_1}{\partial x_1 \partial x_1} \bigg/ \left( \frac{\partial v_1}{\partial x_1} \right) = \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right) + \frac{o}{h\sigma + x_2 q_1} - \frac{(\sigma^2 + 1)}{h(\sigma^2 + 1) + 2x_2(1 + oq_1)}.$$

Use equation (28) to eliminate the last term.

$$\frac{\partial^2 v_1}{\partial x_1 \partial x_1} = \left( \frac{\partial v_1}{\partial x_1} \right) \left\{ \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right) + \frac{o}{h\sigma + x_2 q_1} + \frac{2}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right) + \frac{1}{x_1} \right\}$$

i.e.

$$\frac{\partial^2 v_1}{\partial x_1 \partial x_1} = \frac{3}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right) \left( \frac{\partial v_1}{\partial x_1} \right) + \frac{\partial v_1}{\partial x_1} \left\{ \frac{o}{h\sigma + x_2 q_1} + \frac{1}{x_1} \right\}. \quad (36)$$

Evaluated at the standard cut-off position, the above expression (36) is

$$\frac{\partial^2 v_1}{\partial x_1 \partial x_1} = -0.0007797 \text{ ft/sec per (n. mile)}^2.$$

APPENDIX II

First and Second Differentials of Reference Velocity  
for Least Reference Speed

It is quoted in the text, equations (16), that

$$v_1^2 = \frac{2g R^2}{r_1} \left( \frac{d-h}{r_1+r_2+d} \right).$$

Regard this as the definition of the reference speed  $v_1$  in terms of the out-off position  $(r_1, d)$  where  $r_1$  is the distance from the centre of the earth and  $d$  is the straight line distance from the target. As shown in equation (17),  $d$  may be related to the angular range  $\theta_1$  subtended at the centre of the earth through the equation

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos \theta_1.$$

Differentiate logarithmically with respect to  $r_1$ ;

$$\frac{2\delta v_1}{v_1} = -\frac{\delta r_1}{r_1} + \frac{\delta d - \delta r_1}{d-h} - \frac{\delta d + \delta r_1}{r_1 + r_2 + d}$$

$$\text{i.e.} \quad \frac{2\delta v_1}{v_1} = -\frac{\delta r_1}{r_1} + 2 \cdot \frac{r_1 \delta d - (d+r_2)\delta r_1}{(d-h)(r_1+r_2+d)} \quad (37)$$

using the definition of  $h$  in equation (13).

Differentiate equation (17) with respect to  $r_1$

$$d \cdot \delta d = \delta r_1 (r_1 - r_2 \cos \theta_1). \quad (38)$$

Thus

$$\begin{aligned} 2d \{r_1 \delta d - (d+r_2)\delta r_1\} &= 2\delta r_1 \{r_1^2 - r_1r_2 \cos \theta_1 - d^2 - r_2d\} \\ &= \delta r_1 \{2r_1^2 - r_1^2 - r_2^2 + d^2 - 2d^2 - 2r_2d\} \end{aligned}$$

using equation (17) for  $\cos \theta_1$

$$\begin{aligned} &= \delta r_1 \{r_1^2 - r_2^2 - 2r_2d - d^2\} \\ &= \delta r_1 \{r_1^2 - (r_2+d)^2\} \\ &= -\delta r_1 (d+r_1+r_2)(d-h). \end{aligned}$$

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Substituting in equation (37) gives

$$\delta v_1 = - \frac{v_1 \delta r_1}{2} \left\{ \frac{1}{r_1} + \frac{1}{d} \right\}. \quad (39)$$

At the standard out-off position defined in Appendix I it may be shown that

$$d = 2482.60 \text{ n. miles.}$$

Evaluated at the standard out-off position, the above expression (39) is

$$\frac{\partial v_1}{\partial r_1} = - 6.20203 \text{ ft/sec per n. mile of height.}$$

When the ground range to the out-off point varies, equation (17) shows that  $d$  varies but  $r_1$  remains constant.

Differentiate equation (17) with respect to  $\phi_1$

$$d \cdot \delta d = r_1 r_2 \sin \phi_1 \cdot \delta \phi_1.$$

From equation (31) it follows that

$$\delta d = - \frac{r_1 r_2}{Rd} \sin \phi_1 \cdot \delta x_1 \quad (40)$$

where  $\delta x_1$  is the change in ground range to the cut-off position.

Differentiate equation (16) logarithmically with respect to  $d$ ;

$$\begin{aligned} \frac{2\delta v_1}{v_1} &= \frac{\delta d}{d-h} - \frac{\delta d}{r_1+r_2+d} \\ &= \frac{\delta d \cdot 2r_1}{(d-h)(r_1+r_2+d)}. \end{aligned}$$

Using equation (40) for  $\delta d$ , this becomes

$$\frac{\partial v_1}{\partial x_1} = \frac{r_1 v_1}{(d-h)(r_1+r_2+d)} \cdot \left( - \frac{r_1 r_2}{Rd} \sin \phi_1 \right). \quad (41)$$

Evaluated at the standard out-off position, the above expression (41) is

$$\frac{\partial v_1}{\partial x_1} = - 2.691264 \text{ ft/sec per n. mile ground range.}$$

Equation (39) may be written

$$\frac{\partial v_1}{\partial x_1} = - \frac{v_1 (r_1 + d)}{2r_1 d}.$$

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Differentiate logarithmically once more with respect to  $r_1$ ;

$$\begin{aligned}\frac{\partial^2 v_1}{\partial x_1^2} \bigg/ \frac{\partial v_1}{\partial x_1} &= \frac{1}{v_1} \cdot \frac{\partial v_1}{\partial x_1} + \frac{1 + \frac{\partial d}{\partial x_1}}{r_1 + d} - \frac{1}{r_1} - \frac{1}{d} \left( \frac{\partial d}{\partial x_1} \right) \\ &= \frac{1}{v_1} \cdot \frac{\partial v_1}{\partial x_1} - \frac{d}{r_1(r_1 + d)} - \frac{r_1}{d(r_1 + d)} \cdot \frac{\partial d}{\partial x_1}.\end{aligned}$$

Using equation (38),

$$\begin{aligned}\frac{\partial^2 v_1}{\partial x_1^2} &= \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 - \left( \frac{\partial v_1}{\partial x_1} \right) \frac{1}{r_1 d (r_1 + d)} \left\{ d^2 + \frac{r_1^2}{d} (r_1 - r_2 \cos \phi_1) \right\} \\ &= \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 - \left( \frac{\partial v_1}{\partial x_1} \right) \frac{d^3 + r_1^3 - r_1^2 r_2 \cos \phi_1}{r_1 d^2 (r_1 + d)}\end{aligned}$$

which becomes on substituting for  $\cos \phi_2$  from equation (17)

$$\begin{aligned}&= \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 - \left( \frac{\partial v_1}{\partial x_1} \right) \frac{2d^3 + 2r_1^3 - r_1^3 - r_1^2 r_2 + r_1 d^2}{2r_1 d^2 (r_1 + d)} \\ &= \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 - \left( \frac{\partial v_1}{\partial x_1} \right) \frac{2d^2(r_1 + d) + r_1^3 - r_1^2 r_2 - r_1 d^2}{2r_1 d^2 (r_1 + d)}\end{aligned}$$

$$\text{i.e.} \quad \frac{\partial^2 v_1}{\partial x_1^2} = \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 - \frac{1}{r_1} \left( \frac{\partial v_1}{\partial x_1} \right) + \frac{v_1 (r_1^2 - r_2^2 - d^2)}{4r_1 d^3}. \quad (42)$$

Evaluated at the standard cut-off position, the above expression (42) is

$$\frac{\partial^2 v_1}{\partial x_1^2} = 0.003422 \text{ ft/sec per (n. mile)}^2.$$

Equation (41) may be written

$$\frac{\partial v_1}{\partial x_1} = - \frac{v_1 r_1^2 r_2 \sin \phi_1}{Rd(d-h)(r_1 + r_2 + d)}.$$

Differentiate logarithmically once more with respect to  $x_1$

$$\frac{\partial^2 v_1}{\partial x_1^2} \bigg/ \frac{\partial v_1}{\partial x_1} = \frac{1}{v_1} \cdot \frac{\partial v_1}{\partial x_1} + \cot \phi_1 \frac{\partial \phi_1}{\partial x_1} - \frac{\partial d}{\partial x_1} \left\{ \frac{1}{d} + \frac{1}{d-h} + \frac{1}{r_1 + r_2 + d} \right\}$$

which becomes on using equations (31) and (40)



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$$\frac{\partial^2 v_1}{\partial x_1^2} \frac{\partial v_1}{\partial x_1} = \frac{1}{v_1} \cdot \frac{\partial v_1}{\partial x_1} - \frac{1}{R} \cot \Phi_1 + \frac{r_1 r_2}{Rd} \sin \Phi_1 \left\{ \frac{1}{d} + \frac{2(d+r_2)}{(d-h)(r_1+r_2+d)} \right\} \quad (43)$$

Last term

$$\frac{r_1 r_2}{Rd} \sin \Phi_1 \left\{ \frac{1}{d} + \frac{2(d+r_2)}{(d-h)(r_1+r_2+d)} \right\}$$

may be written with the use of equations (13) and (41) as

$$\begin{aligned} & - \frac{1}{r_1 v_1} \left( \frac{\partial v_1}{\partial x_1} \right) \frac{(r_1+r_2+d)(d-r_1+r_2) + 2d(r_2+d)}{d} \\ & = - \left( \frac{\partial v_1}{\partial x_1} \right) \frac{(d+r_2)^2 + 2d(r_2+d) - r_1^2}{r_1 v_1 d} \\ & = - \left( \frac{\partial v_1}{\partial x_1} \right) \frac{(2d+r_2)^2 - r_1^2 - d^2}{r_1 v_1 d} . \end{aligned}$$

Substituting in equation (43) gives

$$\frac{\partial^2 v_1}{\partial x_1^2} = \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right)^2 - \frac{\cot \Phi_1}{R} \left( \frac{\partial v_1}{\partial x_1} \right) - \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right) \left\{ \frac{(2d+r_2)^2 - r_1^2 - d^2}{r_1 d} \right\} \quad (44)$$

Evaluated at the standard cut-off position, the above expression (44) is

$$\frac{\partial^2 v_1}{\partial x_1^2} = -0.001087 \text{ ft/sec per (n. mile)}^2 .$$

Again equation (39) may be written

$$\frac{\partial v_1}{\partial x_1} = - \frac{v_1(r_1+d)}{2r_1 d} .$$

Differentiate logarithmically with respect to  $x_1$  keeping  $r_1$  constant;

$$\frac{\partial^2 v_1}{\partial x_1 \partial x_1} \frac{\partial v_1}{\partial x_1} = \frac{1}{v_1} \cdot \frac{\partial v_1}{\partial x_1} + \frac{\partial d}{\partial x_1} \left\{ \frac{1}{r_1+d} - \frac{1}{d} \right\}$$

and so using equation (40)

$$= \frac{1}{v_1} \cdot \frac{\partial v_1}{\partial x_1} + \frac{r_1 r_2 \sin \Phi_1}{Rd} \cdot \frac{r_1}{d(r_1+d)}$$

$$\text{i.e.} \quad \frac{\partial^2 v_1}{\partial x_1 \partial r_1} = \frac{1}{v_1} \left( \frac{\partial v_1}{\partial x_1} \right) \left( \frac{\partial v_1}{\partial r_1} \right) + \left( \frac{\partial v_1}{\partial r_1} \right) \frac{r_1^2 r_2 \sin \phi_1}{Rd^2 (r_1 + d)}. \quad (45)$$

Evaluated at the standard out-off position, the above expression (45) is

$$\frac{\partial^2 v_1}{\partial x_1 \partial r_1} = -0.0004683 \text{ ft/sec per (n. mile)}^2.$$

It is quoted in the text, equation (18), that the climb angle  $\theta_1$  is given by

$$\cot 2\theta_1 = \frac{r_1 - r_2 \cos \phi_1}{r_2 \sin \phi_1}.$$

Differentiate with respect to  $r_1$  treating  $\phi_2$  as constant

$$-\operatorname{cosec}^2 2\theta_1 \cdot 2d\theta_1 = \frac{dr_1}{r_2 \sin \phi_1}$$

$$\therefore \frac{\partial \theta_1}{\partial r_1} = -\frac{1}{2r_2 \sin \phi_1 \operatorname{cosec}^2 2\theta_1}$$

and by substitution from equation (18) this is

$$= -\frac{1}{2} \cdot \frac{r_2 \sin \phi_1}{r_2^2 \sin^2 \phi_1 + (r_1 - r_2 \cos \phi_1)^2}$$

$$\text{i.e.} \quad \frac{\partial \theta_1}{\partial r_1} = -\frac{r_2 \sin \phi_1}{2d^2} \quad (46)$$

using equation (17) for  $d^2$ .

Evaluated at the standard out-off position, the above expression (46) is

$$\frac{\partial \theta_1}{\partial r_1} = -0.6374 \text{ minutes of arc per n. mile.}$$

Differentiating equation (18) with respect to ground range gives

$$-\operatorname{cosec}^2 2\theta_1 \cdot \frac{2\partial \theta_1}{\partial \phi_2} = \frac{(r_2 \sin \phi_1)^2 - (r_1 - r_2 \cos \phi_1) r_2 \cos \phi_1}{(r_2 \sin \phi_1)^2}$$

which using equations (18) and (17) leads to

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$$\frac{\partial \theta_1}{\partial \phi_2} = -\frac{r_2}{2d^2} (r_2 \sin^2 \phi_1 - r_1 \cos \phi_1 + r_2 \cos^2 \phi_1) .$$

Hence, from equation (31) it follows that

$$\frac{\partial \theta_1}{\partial x_1} = \frac{r_2}{2Rd^2} (r_2 - r_1 \cos \phi_1) . \quad (47)$$

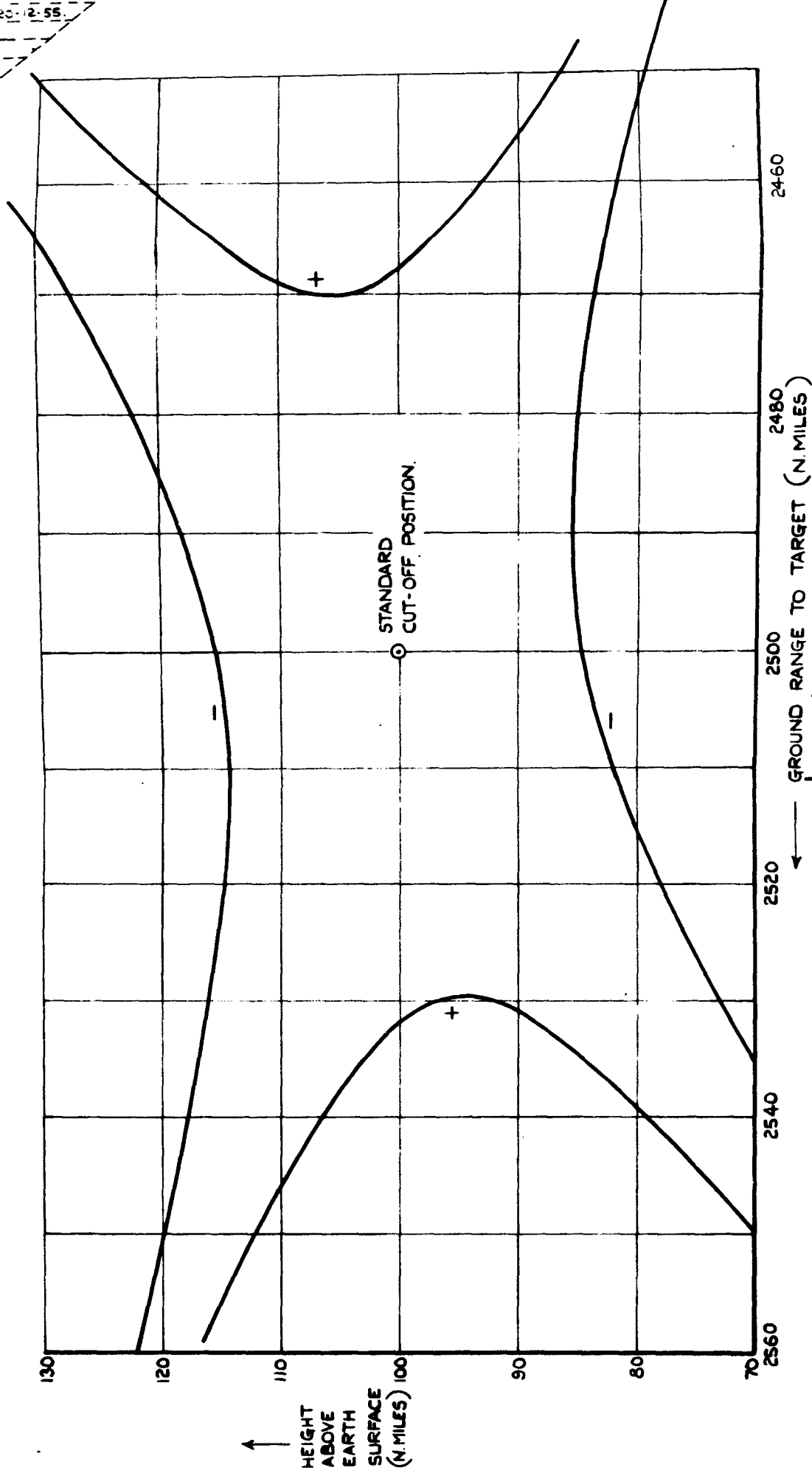
Evaluated at the standard cut-off position, the above expression (47) is

$$\frac{\partial \theta_1}{\partial x_1} = 0.2217 \text{ minutes of arc per n. mile.}$$

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**FIG.1. CONTOUR OF ERROR OF  $\frac{1}{2}$  FT/SEC. BETWEEN LINEAR APPROXIMATION TO REFERENCE SPEED AND ACCURATE REFERENCE SPEED.**  
 REFERENCE SPEED DEFINED AS SPEED TO REACH TARGET GIVEN CLIMB ANGLE  $33\frac{1}{2}$  DEGREES.

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FIG. 2.

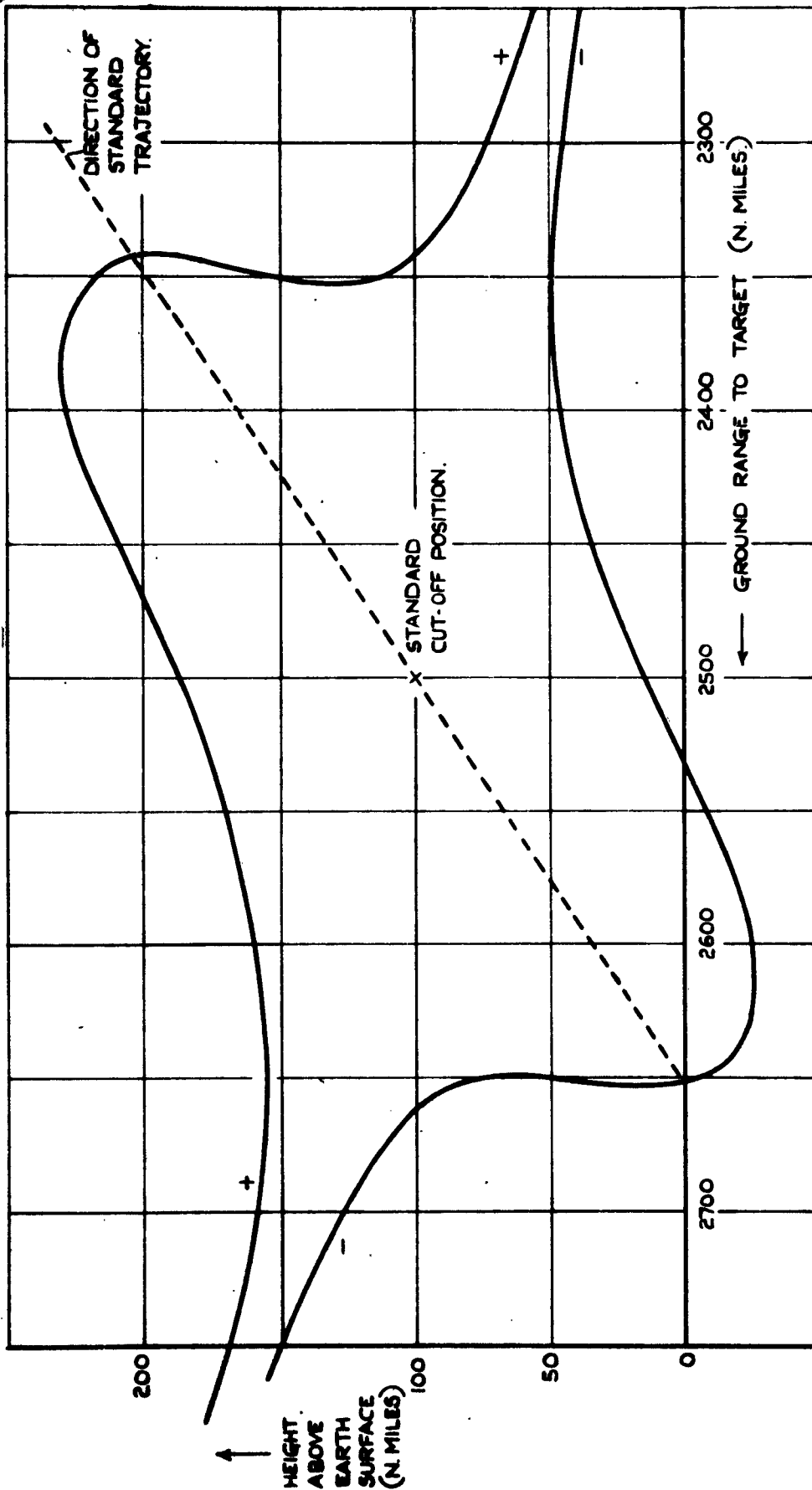


FIG. 2. CONTOUR OF ERROR OF  $\frac{1}{2}$  FT/SEC. BETWEEN QUADRATIC APPROXIMATION  
 TO REFERENCE SPEED AND ACCURATE REFERENCE SPEED.

CLIMB ANGLE ASSUMED CONSTANT AT  $33\frac{1}{2}$  DEGREES.  
 TRAJECTORY TO TARGET ASSUMED TO BE VACUUM ORBIT ABOUT SPHERICAL NON-ROTATING EARTH.



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